# CATEGORIFICATION OF THE VIRTUAL BRAID GROUPS

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ABSTRACT. Using complexes of Soergel bimodules, we extend Rouquier's categorification of the braid groups to the virtual braid groups.

KEYWORDS: braid group, virtual braid, categorification

## INTRODUCTION

Virtual links have been introduced by Kauffman in [5] as a geometric counterpart of Gauss codes. A virtual knot diagram is a generic oriented immersion of circles into the plane, with the usual positive and negative crossings plus a new kind of crossings called virtual. Such crossings appear for instance when one projects a generic link in a thickened surface onto a plane (see [2] or [7]). Many invariants for classical links can be extended to virtual links. Classical oriented links can be represented by closed braids; likewise virtual links can be represented by the closures of virtual braids. There is also an analogue of Markov theorem [3]. Now virtual braids with $n$ strands form a group, denoted $\mathcal{VB}_n$, which can be described by generators and relations, generalizing the generators and relations of the usual braid group with $n$ strands $\mathcal{B}_n$. The aim of this note is, using this presentation, to categorify $\mathcal{VB}_n$ in the weak sense of Rouquier. Rouquier actually proves in [10] a stronger version of the result that we want to extend and in particular he states the faithfulness of his categorification. More precisely, to any word $\omega$ in the generators of $\mathcal{VB}_n$ we associate a bounded cochain complex $F(\omega)$ of bimodules such that if two words $\omega$ and $\omega'$ represent the same element of $\mathcal{VB}_n$, then the corresponding cochain complexes $F(\omega)$ and $F(\omega')$ are homotopy equivalent. So we obtain a morphism from the group $\mathcal{VB}_n$ to the group of isomorphim classes of invertible complexes up to homotopy. This morphism, contrary to the case of $\mathcal{B}_n$ studied by Rouquier, is not injective. We describe a part of its kernel in Remarks 3.5.

Note that, using Mazorchuk and Stroppel's study of Arkhipov's twisting functor in [9], one can give a representation theoretic approach to the categorification of virtual braids.

## 1. VIRTUAL BRAIDS

Following [13] and [8], we recall the definition of the virtual braid group $\mathcal{VB}_n$ with $n$ strands.

**Definition 1.1.** *The virtual braid group $\mathcal{VB}_n$ is the group generated by $2(n-1)$ generators $\sigma_1, \ldots, \sigma_{n-1}$ and $\zeta_1, \ldots, \zeta_{n-1}$ satisfying the braid group relations*

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad if \ |i - j| > 1, \tag{1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad if \ 1 \leq i \leq n - 2, \tag{2}$$

*the permutation group relations*

$$\zeta_i\zeta_j = \zeta_j\zeta_i, \quad if\ |i-j| > 1, \tag{3}$$

$$\zeta_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\zeta_{i+1}, \quad if\ 1 \le i \le n-2, \tag{4}$$

$$\zeta_i^2 = 1, \quad if\ 1 \le i \le n-1, \tag{5}$$

*and the mixed relations*

$$\sigma_i\zeta_j = \zeta_j\sigma_i, \quad if\ |i-j| > 1, \tag{6}$$

$$\sigma_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\sigma_{i+1}, \quad if\ 1 \le i \le n-2. \tag{7}$$

The classical braid group $\mathcal{B}_n$ (see [4] for a definition) naturally embeds in $\mathcal{VB}_n$ as a subgroup generated by $\sigma_1, \ldots, \sigma_{n-1}$.

The braid group $\mathcal{VB}_n$ can be depicted diagrammatically. To each generator $\sigma_i$ (resp. $\sigma_i^{-1}$) we associate the elementary braid diagram consisting of a single positive (resp. negative) crossing between the $i$th and $i+1$st strand as shown in Figure 1 (resp. Figure 2). To each generator $\zeta_i$ we associate the elementary virtual braid diagram with a single virtual crossing between the $i$th and $i+1$st strand of Figure 3.
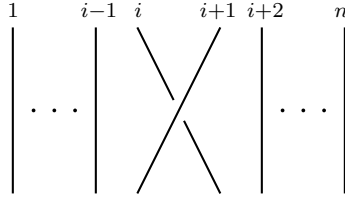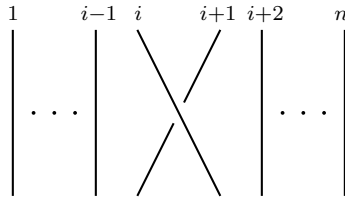


FIGURE 1. The positive braid $\sigma_i$



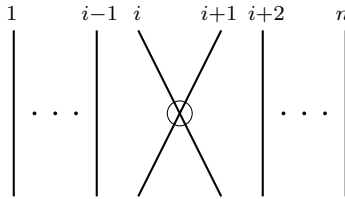FIGURE 2. The negative braid $\sigma_i^{-1}$



FIGURE 3. The virtual braid $\zeta_i$

The multiplication law of the group $\mathcal{VB}_n$ consists in concatenating these elementary braids. We use the convention that braids multiply from bottom to top: if $D$ (resp. $D'$) is a virtual braid diagram representing an element $\beta$ (resp. $\beta'$) of $\mathcal{VB}_n$, then the product $\beta\beta'$ is represented by the diagram obtained by putting $D'$ on top of $D$ and gluing the lower endpoints of $D'$ to the upper endpoints of $D$.

The braid group relations, the permutation group relations and the mixed relations have a diagrammatical interpretation. They correspond to planar isotopies and the generalized Reidemeister moves depicted in Figures 4, 5 and 6.
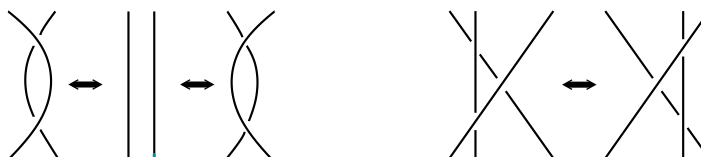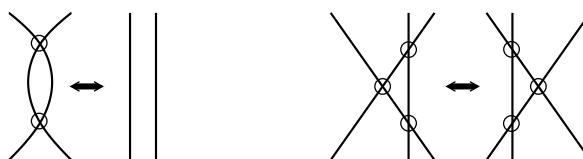


FIGURE 4. Classical Reidemeister II–III moves



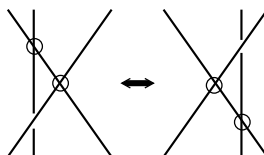FIGURE 5. Virtual Reidemeister moves



FIGURE 6. The mixed Reidemeister move

## 2. ROUQUIER'S CATEGORIFICATION OF THE BRAID GROUPS

In this section we recall how Rouquier [10] categorified the braid group $\mathcal{B}_n$.

2.1. **Soergel bimodules.** We first discuss some bimodules introduced by Soergel [11], [12] in his work on representation theory.

Let $R$ be the subalgebra of $\mathbb{Q}[x_1, \ldots, x_n]$ defined by

$$R = \mathbb{Q}[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n] = \mathbb{Q}[x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n].$$

The symmetric group $S_n$ acts on $\mathbb{Q}[x_1, \ldots, x_n]$ by $\omega(x_i) = x_{\omega(i)}$ for all $x_i \in R$ and $\omega \in S_n$. This action preserves $R$. Let $R^\omega$ be the subalgebra

of elements of $R$ fixed by $\omega$. In particular $R^{\tau_i}$ is the subalgebra of $R$ of elements fixed by the transposition $\tau_i = (i, i+1)$. As an algebra,

$$R^{\tau_i} = \mathbb{Q}[x_1 - x_2, \ldots, (x_1 - x_i) + (x_1 - x_{i+1}),$$
$$(x_1 - x_i)(x_1 - x_{i+1}), x_1 - x_{i+2}, \ldots, x_1 - x_n].$$

Let us also consider the $R$–bimodules $B_\omega = R \otimes_{R^\omega} R$ for any $\omega \in S_n$. The $R$–bimodules $B_{\tau_i}$ will be denoted by $B_i$ for simplicity of notation. We introduce a grading on $R$, $R^{\tau_i}$ and $B_i$ by setting $\deg(x_k) = 2$ for all $k = 1, \ldots, n$.

Two $R$-bimodule morphisms between these objects will be relevant to us, namely $\mathrm{br}_i : B_i \to R$ and $\mathrm{rb}_i : R\{2\} \to B_i$ defined by

$$\mathrm{br}_i(1 \otimes 1) = 1 \quad \text{and} \quad \mathrm{rb}_i(1) = (x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1}).$$

The curly brackets indicate a shift of the grading: if $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a $\mathbb{Z}$-graded bimodule and $p$ an integer, then the $\mathbb{Z}$–graded bimodule $M\{p\}$ is defined by $M\{p\}_i = M_{i-p}$ for all $i \in \mathbb{Z}$. The maps $\mathrm{br}_i$ and $\mathrm{rb}_i$ are degree-preserving morphisms of graded $R$-bimodules.

## 2.2. Categorification of the braid groups.

Following [6] and [10], to each braid generator $\sigma_i \in \mathcal{B}_n$ we assign the cochain complex $F(\sigma_i)$ of graded $R$–bimodules

$$F(\sigma_i) : 0 \longrightarrow R\{2\} \xrightarrow{\mathrm{rb}_i} B_i \longrightarrow 0, \tag{8}$$

where $B_i$ sits in cohomological degree 0. To $\sigma_i^{-1}$ we assign the cochain complex $F(\sigma_i^{-1})$ of graded $R$–bimodules

$$F(\sigma_i^{-1}) : 0 \longrightarrow B_i\{-2\} \xrightarrow{\mathrm{br}_i} R\{-2\} \longrightarrow 0, \tag{9}$$

where $B_i\{-2\}$ sits in cohomological degree 0. To the unit element $1 \in \mathcal{B}_n$ we assign the complex of graded $R$-bimodules

$$F(1) : 0 \longrightarrow R \longrightarrow 0, \tag{10}$$

where $R$ sits in cohomological degree 0; the complex $F(1)$ is a unit for the tensor product of complexes so tensoring any complex of graded $R$-bimodules with $F(1)$ leaves the complex unchanged. Finally to any word $\sigma = \sigma_{i_1}^{\varepsilon_1} \ldots \sigma_{i_k}^{\varepsilon_k}$ where $\varepsilon_1, \ldots, \varepsilon_k = \pm 1$, we assign the complex of graded $R$-bimodules $F(\sigma) = F(\sigma_{i_1}^{\varepsilon_1}) \otimes_R \ldots \otimes_R F(\sigma_{i_k}^{\varepsilon_k})$.

Rouquier proved the following result, which can be called a categorification of the braid group $\mathcal{B}_n$.

**Theorem 2.1.** [10] *If $\omega$ and $\omega'$ are words representing the same element of $\mathcal{B}_n$, then $F(\omega)$ and $F(\omega')$ are homotopy equivalent complexes of $R$–bimodules.*

## 3. CATEGORIFICATION OF THE VIRTUAL BRAID GROUPS

Our aim is to extend Rouquier's categorification to the virtual braid groups $\mathcal{VB}_n$. The cochain complexes associated to the generators $\sigma_i$ of $\mathcal{VB}_n$ coming from $\mathcal{B}_n$ will be the same as Rouquier's complexes above. We have to assign complexes to the generators $\zeta_i$ of $\mathcal{VB}_n$ corresponding to virtual crossings such that all these complexes satisfy the same relations as the generators of $\mathcal{VB}_n$ up to homotopy equivalence.

3.1. **Twisted bimodules.** In order to achieve this categorification, we consider the $R$-bimodule $R_\omega$ for each permutation $\omega \in S_n$. As a left $R$-module, $R_\omega$ is equal to $R$ while the right action of $a \in R$ is the multiplication by $\omega(a)$. Note that $R_{\mathrm{id}} = R$ as an $R$-bimodule. The following lemma is obvious.

**Lemma 3.1.** *For all $\omega, \omega' \in S_n$ the map $\psi : R_\omega \otimes_R R_{\omega'} \to R_{\omega\omega'}$ defined for all $a, b \in R$ by $\psi(a \otimes b) = a\omega(b)$ is an isomorphism of $R$-bimodules.*

The bimodules we will mostly use are the bimodules $R_i = R_{\tau_i}$. The reason why we consider these bimodules is that, by Lemma 3.1, they possess the following interesting property:

$$R_i \otimes_R R_i \cong R$$

for all $i = 1, \ldots, n - 1$.

**Lemma 3.2.** *For all permutations $\omega, \omega' \in S_n$ the $R$-bimodules $R_\omega \otimes_R B_{\omega'}$ and $B_{\omega\omega'\omega^{-1}} \otimes_R R_\omega$ are isomorphic.*

*Proof.* First note that there are natural isomorphisms of $R$-bimodules:

$$R_\omega \otimes_R B_{\omega'} \cong R_\omega \otimes_{R^{\omega'}} R \quad \text{and} \quad B_{\omega\omega'\omega^{-1}} \otimes_R R_\omega \cong R \otimes_{R^{\omega\omega'\omega^{-1}}} R_\omega.$$

Now consider the map $\psi : R_\omega \otimes_{R^{\omega'}} R \to R \otimes_{R^{\omega\omega'\omega^{-1}}} R_\omega$ defined for all $a, b \in R$ by

$$\psi(a \otimes b) = a \otimes \omega(b).$$

This map is well defined: for any $c \in R^{\omega'}$, we just have to check that $a \otimes cb$ and $a\omega(c) \otimes b$ have the same image under $\psi$. This is true because $c \in R^{\omega'}$ implies that $\omega(c) \in R^{\omega\omega'\omega^{-1}}$. Moreover the map $\psi$ is obviously a morphism of $R$-bimodules.

Similarly, the map $\varphi : R \otimes_{R^{\omega\omega'\omega^{-1}}} R_\omega \to R_\omega \otimes_{R^{\omega'}} R$ defined for all $a, b \in R$ by

$$\varphi(a \otimes b) = a \otimes \omega^{-1}(b)$$

is a well defined morphism of $R$-bimodules as well.

Finally, $\psi$ and $\varphi$ are easily seen to be inverse of each other. $\square$

Now let us assign a complex of graded $R$-bimodules to each generator of the virtual braid group. To the elements $\sigma_i$ and $\sigma_i^{-1}$ we assign the complexes $F(\sigma_i)$ and $F(\sigma_i^{-1})$ defined by (8) and (9). To the element $\zeta_i$ we assign the complex concentrated in degree 0:

$$F(\zeta_i) : 0 \longrightarrow R_i \longrightarrow 0. \tag{11}$$

Just as in Section 2.2 we assign to the unit element 1 of $\mathcal{VB}_n$ the complex $F(1)$ of (10), and to a virtual braid word we assign the tensor product over $R$ of the complexes associated to the generators involved in the expression of the word.

**Remark 3.3.** Consider $\omega = \zeta_{i_1} \ldots \zeta_{i_k}$ a word in $\{\zeta_1, \ldots, \zeta_{n-1}\}$ and let $\tilde{\omega} = \tau_{i_1} \ldots \tau_{i_k}$ be the corresponding element of $S_n$. It follows from Lemma 3.1 that the complex $F(\omega)$ is isomorphic to $0 \longrightarrow R_{\tilde{\omega}} \longrightarrow 0$.

3.2. **Categorification of $\mathcal{VB}_n$.** We now state our main result.

**Theorem 3.4.** *If $\omega$ and $\omega'$ are words representing the same element of $\mathcal{VB}_n$, then $F(\omega)$ and $F(\omega')$ are homotopy equivalent complexes of $R$–bimodules.*

*Proof.* By definition of $\mathcal{VB}_n$ and in view of Theorem 2.1, it is enough to check that there are homotopy equivalences between the complexes associated to the braid words appearing in both sides of Relations (3)-(7). Actually we will prove a stronger result: these complexes are isomorphic.

*Permutation group relations.* Let $\omega = \omega'$ be a permutation group relation. Since the bimodules $R_{\widetilde{\omega}}$ and $R_{\widetilde{\omega'}}$ are equal, the complexes $F(\omega)$ and $F(\omega')$ are isomorphic in view of Remark 3.3.

*Mixed relations.* Let us first deal with Relation (6). We have to prove that for $|i - j| > 1$ the complexes $F(\zeta_j \sigma_i)$ and $F(\sigma_i \zeta_j)$ are isomorphic. We have

$$F(\zeta_j \sigma_i) : 0 \longrightarrow R_j \otimes_R R\{2\} \xrightarrow{\mathrm{id} \otimes \mathrm{rb}_i} R_j \otimes_R B_i \longrightarrow 0$$

and

$$F(\sigma_i \zeta_j) : 0 \longrightarrow R \otimes_R R_j\{2\} \xrightarrow{\mathrm{rb}_i \otimes \mathrm{id}} B_i \otimes_R R_j \longrightarrow 0.$$

First observe that $F(\zeta_j \sigma_i)$ is naturally isomorphic to the following complex of bimodules

$$0 \longrightarrow R_j\{2\} \xrightarrow{d} R_j \otimes_{R^{\tau_i}} R \longrightarrow 0$$

whose differential $d$ sends each $a \in R_j\{2\}$ to

$$a\big(\tau_j(x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})\big) = a\big((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})\big).$$

Similarly, $F(\sigma_i \zeta_j)$ is isomorphic to

$$0 \longrightarrow R_j\{2\} \xrightarrow{d'} R \otimes_{R^{\tau_i}} R_j \longrightarrow 0$$

whose differential $d'$ sends each $a \in R_j\{2\}$ to

$$a\big((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})\big).$$

Since the transpositions $\tau_i$ and $\tau_j$ commute, the proof of Lemma 3.2 provides us the isomorphism of $R$-bimodules $\psi : R_j \otimes_{R^{\tau_i}} R \to R \otimes_{R^{\tau_i}} R_j$. Using the invariance of $(x_i - x_{i+1})$ under the action of $\tau_j$, we easily check that the following vertical maps and their inverse commute with the differentials.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & R_j\{2\} & \xrightarrow{d} & R_j \otimes_{R^{\tau_i}} R & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & R_j\{2\} & \xrightarrow{d'} & R \otimes_{R^{\tau_i}} R_j & \longrightarrow & 0
\end{array}
$$

Thus the complexes $F(\zeta_j \sigma_i)$ and $F(\sigma_i \zeta_j)$ are isomorphic for all $|i - j| > 1$.

We finally deal with Relation (7). We have to show that the complexes $F(\zeta_{i+1} \zeta_i \sigma_{i+1})$ and $F(\sigma_i \zeta_{i+1} \zeta_i)$ are isomorphic for $i = 1, \ldots, n - 2$. The complex $F(\zeta_{i+1} \zeta_i \sigma_{i+1})$ is equal to

$$0 \longrightarrow R_{i+1} \otimes_R R_i \otimes_R R\{2\} \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_{i+1}} R_{i+1} \otimes_R R_i \otimes_R B_{i+1} \longrightarrow 0$$

and the complex $F(\sigma_i \zeta_{i+1} \zeta_i)$ is equal to

$$0 \longrightarrow R \otimes_R R_{i+1} \otimes_R R_i\{2\} \xrightarrow{\mathrm{rb}_i \otimes \mathrm{id} \otimes \mathrm{id}} B_i \otimes_R R_{i+1} \otimes_R R_i \longrightarrow 0.$$

There exist a natural isomorphism between $F(\zeta_{i+1} \zeta_i \sigma_{i+1})$ and the complex

$$0 \longrightarrow R_{\tau_{i+1}\tau_i}\{2\} \xrightarrow{d} R_{\tau_{i+1}\tau_i} \otimes_{R^{\tau_{i+1}}} R \longrightarrow 0$$

whose differential $d$ sends each $a \in R_{\tau_{i+1}\tau_i}\{2\}$ to

$$a\big(\tau_{i+1}\tau_i(x_{i+1} - x_{i+2}) \otimes 1 + 1 \otimes (x_{i+1} - x_{i+2})\big) =$$
$$a\big((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_{i+1} - x_{i+2})\big).$$

Similarly, $F(\sigma_i \zeta_{i+1} \zeta_i)$ is isomorphic to

$$0 \longrightarrow R_{\tau_{i+1}\tau_i}\{2\} \xrightarrow{d'} R \otimes_{R^{\tau_i}} R_{\tau_{i+1}\tau_i} \longrightarrow 0$$

whose differential $d'$ sends each $a \in R_{\tau_{i+1}\tau_i}\{2\}$ to

$$a\big((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})\big).$$

Applying Lemma 3.2 and using the relation $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ and the involutivity of $\tau_{i+1}$ yields the isomorphism

$$\psi : R_{\tau_{i+1}\tau_i} \otimes_{R^{\tau_{i+1}}} R \to R \otimes_{R^{\tau_i}} R_{\tau_{i+1}\tau_i}$$

given for all $a \in R_{\tau_{i+1}\tau_i}$, $b \in R$ by

$$\psi(a \otimes b) = a \otimes \tau_{i+1}\tau_i(b).$$

Let us complete the proof by checking that the following vertical maps commute with the differentials (and similarly for their inverse).

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & R_{\tau_{i+1}\tau_i}\{2\} & \xrightarrow{d} & R_{\tau_{i+1}\tau_i} \otimes_{R^{\tau_{i+1}}} R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & R_{\tau_{i+1}\tau_i}\{2\} & \xrightarrow{d'} & R \otimes_{R^{\tau_i}} R_{\tau_{i+1}\tau_i} & \longrightarrow & 0
\end{array}
$$

For any element $a$ in $R_{\tau_{i+1}\tau_i}\{2\}$ we compute both its image under $d' \circ \mathrm{id}$ and $\psi \circ d$. We obtain

$$d' \circ \mathrm{id}(a) = a\big((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})\big)$$

and

$$
\begin{aligned}
\psi \circ d(a) &= \psi\big(a(x_i - x_{i+1}) \otimes 1 + a \otimes (x_{i+1} - x_{i+2})\big) \\
&= a(x_i - x_{i+1}) \otimes 1 + a \otimes \tau_{i+1}\tau_i(x_{i+1} - x_{i+2}) \\
&= a\big((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})\big)
\end{aligned}
$$

This shows that the complexes $F(\zeta_{i+1} \zeta_i \sigma_{i+1})$ and $F(\sigma_i \zeta_{i+1} \zeta_i)$ are isomorphic for all $i = 1, \ldots, n-2$. $\square$

## Remarks 3.5.

• Let us call *virtualisation moves* the moves consisting in squeezing a classical crossing between two virtual crossings, as shown in Figure 7. We observe that the complexes $F(\zeta_i \sigma_i^\varepsilon \zeta_i)$ and $F(\sigma_i^\varepsilon)$ are isomorphic for all $i = 1, \ldots, n-1$ and $\varepsilon \in \{-1, 1\}$. This is essentially due to the involutivity of $\tau_i$, which implies (*cf.* Lemma 3.1 and Lemma 3.2) that the bimodules $R_i \otimes_R B_i \otimes_R R_i$ and $B_i$ are isomomorphic. Thus our categorification of $\mathcal{VB}_n$ does not detect the virtualisation moves.
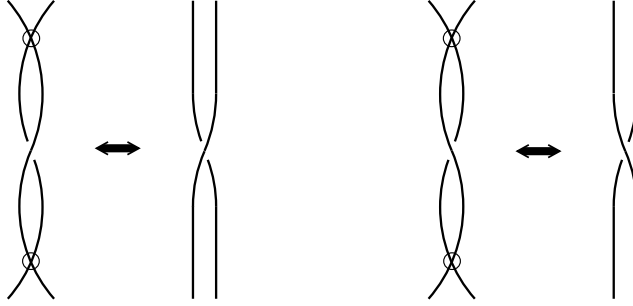
FIGURE 7. Virtualisation moves

• Adding the relation $\zeta_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \zeta_{i+1}$ to the presentation of $\mathcal{VB}_n$, one obtains a presentation of the group of welded braids with $n$ strands defined in [1]. Noting that the only morphism between $R_\omega$ and $R_{\omega'}$ is the trivial one if $\omega \neq \omega'$, one can check that the complexes $F(\zeta_i \sigma_{i+1} \sigma_i)$

$$
\begin{array}{ccc}
& R_i \otimes_R B_{i+1}\{2\} & \\
{\scriptstyle \mathrm{id} \otimes \mathrm{rb}_{i+1}} \nearrow & & \searrow {\scriptstyle \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_i} \\
R_i \otimes_R R\{4\} & & R_i \otimes_R B_{i+1} \otimes_R B_i \\
{\scriptstyle -\,\mathrm{id} \otimes \mathrm{rb}_i} \searrow & & \nearrow {\scriptstyle \mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}} \\
& R_i \otimes_R B_i\{2\} &
\end{array}
$$

and $F(\sigma_{i+1}\sigma_i \rho_{i+1})$

$$
\begin{array}{ccc}
& B_{i+1} \otimes_R R_{i+1}\{2\} & \\
{\scriptstyle \mathrm{rb}_{i+1} \otimes \mathrm{id}} \nearrow & & \searrow {\scriptstyle \mathrm{id} \otimes \mathrm{rb}_i \otimes \mathrm{id}} \\
R \otimes_R R_{i+1}\{4\} & & B_{i+1} \otimes_R B_i \otimes_R R_{i+1} \\
{\scriptstyle -\,\mathrm{rb}_i \otimes \mathrm{id}} \searrow & & \nearrow {\scriptstyle \mathrm{rb}_{i+1} \otimes \mathrm{id} \otimes \mathrm{id}} \\
& B_i \otimes_R R_{i+1}\{2\} &
\end{array}
$$

are not equivalent up to homotopy.

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